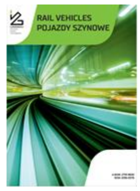
Article citation info:
Magnucki K. Free flexural vibrations of standard wide-flange H-beams with consideration of the shear effect. Rail Vehicles/Pojazdy Szynowe. 0000;xx:xx-xx. https://doi.org/10.53502/RAIL-189244

Rail Vehicles/Pojazdy Szynowe

www.railvehicles.eu



Free flexural vibrations of standard wide-flange H-beams with consideration of the shear effect

Krzysztof **Magnucki** [a, *]

[a] *Łukasiewicz Research Network – Poznań Institute of Technology, Poland*

ARTICLE INFO

Received: 18 March 2024
Revised: 3 April 2024
Accepted: 24 May 2024
Available online: 25 May 2024

KEYWORDS

H-steel beams
Zhuravsky shear stress
Shear deformation theory
Analytical studies
Fundamental natural frequency

The subject of the paper is a simply supported standard wide-flange H-beam. Cross sections of this beam is analytically described as a three-layer structure. The shear effect in its successive layers is taking into account with consideration of the classical shear stress formula called Zhuravsky shear stress. Based on Hamilton's principle, two differential equations of motion are obtained. These equations are analytically solved and the fundamental natural frequency of flexural vibration for this beam is derived. Exemplary calculations are carried out for selected five I-beams.

1. Introduction

The theory of the shear effect occurring in structures, initiated by S.P. Timoshenko in 1921, is intensively improved today. Aghababaei and Reddy [1] improved the earlier one from 1984 third-order shear deformation plate theory and analytically studied the bending and free vibrations of a simply supported rectangular plate. Ghugal and Sharma [3] analyzed static bending and free flexural vibration problems of thick isotropic beams with consideration of a hyperbolic shear deformation theory. Reddy [10] reformulated the classical and shear deformation beam and plate theories taking into account the Eringen nonlocal differential constitutive relations and the von Kàrmàn nonlinear strains. Thai and Vo [14] developed various higher-order shear deformation beam theories due to bending and free vibration of functionally graded beams. Akgöz and Civalek [2] presented a new size-dependent higher-order shear deformation beam model and analytically studied the bending and free vibration of simply supported micro-beams. Sawant and Dahake [12] analytically described the cantilever beam bending with consideration of the a new hyperbolic shear deformation theory. Xiang [16] analytically studied the free vibration of functionally graded beams using a n-order shear deformation theory. Mahi et al. [8] presented the bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates taking into account a new hyperbolic shear deformation theory. Nguyen et al. [9] proposed a new higher-order shear deformation theory and applied it to the analysis of buckling and free vibration problems of isotropic and functionally graded sandwich beams. Sobhy [14] applied a new accurate four-variable shear deformation theory to the analytically studied the hygrothermal buckling and vibration of functionally graded sandwich plates resting on Winkler–Pasternak elastic foundations. Thai et al. [15] proposed a simple shear deformation theory and applied it to the static bending and free vibration analytically studied of isotropic nanobeams. Magnucki [5] presented analytical models of the sandwich beam and I-beam with consideration of the shear effect, and also static bending studies of these beams. Magnucki et al. [6] Magnucki et al. [6] elaborated the nonlinear shear deformation theory of a beam based on the Zhuravsky shear stress formula and determined

* Corresponding author: krzysztof.magnucki@pit.lukasiewicz.gov.pl (K. Magnucki)

the deflections of sample beams. Ren et al. [11] proposed a new general third-order zigzag model of laminated composite beams and determined analytically the shear stress distribution for example beams. Magnucki [7] presented the individual shear deformation theory and its application in the bending analysis of homogeneous, sandwich and FGM beams. Guo and Shi [4] analyzed the free vibration of laminated composite plates taking into account a refined third-order shear deformation theory.

The subject of the paper is a simply supported standard wide-flange H-beam of length L and depth h (Fig. 1).

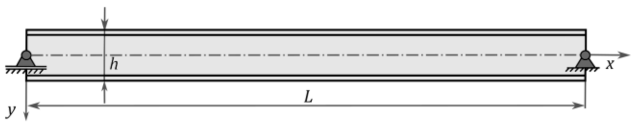


Fig. 1. Scheme of the beam

The cross section of this beam is shown in Fig.2.

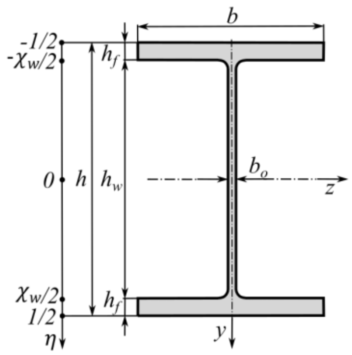


Fig. 2. Scheme of the cross section

Sizes of this cross section are as follows: b – width, h_f – flanges thicknesses, h_w – web depth. However, the dimensionless sizes are as follows: $\chi_w = h_w/h$, $\beta_0 = b_0/b$ and the dimensionless coordinate $\eta = y/h$.

The main goal of the work is to develop the analytical model of this beam and determine its fundamental natural frequency taking into account the shear effect.

2. Analytical model the H-beam

Taking into account the paper [5], the structure of the wide-flange H-beam is analogous to a sandwich beam. Therefore, the following three layers are distinguished (Fig.2):

- the upper flange ($-1/2 \le \eta \le -\chi_w/2$)

$$b(\eta) = b = \text{const} \quad (1)$$

- the web ($-\chi_w/2 \le \eta \le \chi_w/2$)

$$b(\eta) = bf_w(\eta) \quad (2)$$

where

$$f_w(\eta) = \beta_0 + (2\alpha_r - \beta_0)\tan^n\left(\frac{\pi}{2}\frac{\eta}{\chi_w}\right) \quad (3)$$

and: n – even exponent, α_r – dimensionless coefficient

- the lower flange ($\chi_w/2 \le \eta \le 1/2$)

$$b(\eta) = b = \text{const} \quad (4)$$

The deformation of a planar cross section of this beam after bending is graphical presented in Fig. 3.

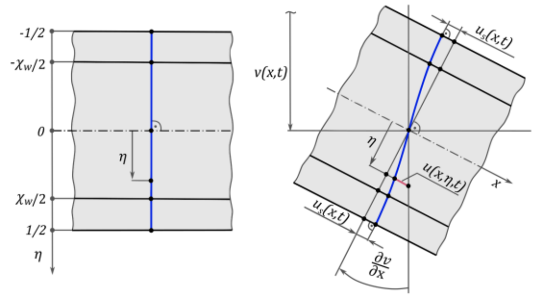


Fig. 3. The deformation scheme of a planar cross section

Thus, the displacements based on the above scheme (Fig. 3) in successive layers are as follows:

- the upper flange ($-1/2 \le \eta \le -\chi_w/2$)

$$u^{(uf)}(x,\eta,t) = -h\left[\eta\frac{\partial v}{\partial x} - f_d^{(uf)}(\eta)\psi_s(x,t)\right] \quad (5)$$

- the web ($-\chi_w/2 \le \eta \le \chi_w/2$)

$$u^{(w)}(x,\eta,t) = -h\left[\eta\frac{\partial v}{\partial x} - f_d^{(w)}(\eta)\psi_s(x,t)\right] \quad (6)$$

- the lower flange ($\chi_w/2 \le \eta \le 1/2$)

$$u^{(lf)}(x,\eta,t) = -h\left[\eta\frac{\partial v}{\partial x} - f_d^{(lf)}\psi_s(\eta)(x,t)\right] \quad (7)$$

where: v(x,t) – deflection, $\psi_s(x,t) = u_s(x,t)/h$ – dimensionless longitudinal displacement function, $f_d^{(uf)}(\eta)$, $f_d^{(w)}(\eta)$, $f_d^{(lf)}(\eta)$ – dimensionless deformation functions, t – time.

Consequently, the strains and stresses – Hooke's law in these layers are in the following form:

- the upper flange ($-1/2 \le \eta \le -\chi_w/2$)

$$\varepsilon_x^{(uf)}(x,\eta,t) = -h\left[\eta\frac{\partial^2 v}{\partial x^2} - f_d^{(uf)}(\eta)\frac{\partial\psi_s}{\partial x}\right] \quad (8)$$

$$\gamma_{xy}^{(uf)}(x,\eta,t) = \frac{df_d^{(uf)}}{d\eta}\psi_s(x,t) \quad (9)$$

$$\sigma_x^{(uf)}(x,\eta,t) = E\varepsilon_x^{(uf)}(x,\eta,t) \quad (10)$$

$$\tau_{xy}^{(uf)}(x,\eta,t) = \frac{E}{2(1+\nu)}\gamma_{xy}^{(uf)}(x,\eta,t) \quad (11)$$

- the web ($-\chi_w/2 \le \eta \le \chi_w/2$)

$$\varepsilon_x^{(w)}(x,\eta,t) = -h\left[\eta\frac{\partial^2 v}{\partial x^2} - f_d^{(w)}(\eta)\frac{\partial\psi_s}{\partial x}\right] \quad (12)$$

$$\gamma_{xy}^{(w)}(x,\eta,t) = \frac{df_d^{(w)}}{d\eta}\psi_s(x,t) \tag{13}$$

$$\sigma_x^{(w)}(x,\eta,t) = E\varepsilon_x^{(w)}(x,\eta,t) \tag{14}$$

$$\tau_{xy}^{(w)}(x,\eta,t) = \frac{E}{2(1+\nu)}\gamma_{xy}^{(w)}(x,\eta,t) \tag{15}$$

- the lower flange ($\chi_w/2 \le \eta \le 1/2$)

$$\varepsilon_x^{(lf)}(x,\eta,t) = -h\left[\eta\frac{\partial^2 v}{\partial x^2} - f_d^{(lf)}(\eta)\frac{\partial\psi_s}{\partial x}\right] \tag{16}$$

$$\gamma_{xy}^{(lf)}(x,\eta,t) = \frac{df_d^{(lf)}}{d\eta}\psi_s(x,t) \tag{17}$$

$$\sigma_x^{(lf)}(x,\eta,t) = E\varepsilon_x^{(lf)}(x,\eta,t) \tag{18}$$

$$\tau_{xy}^{(lf)}(x,\eta,t) = \frac{E}{2(1+\nu)}\gamma_{xy}^{(lf)}(x,\eta,t) \tag{19}$$

where: E – Young's modulus, ν – Poisson ratio.

Taking into account the papers Magnucki et al. [6] and Magnucki [7], the unknown dimensionless deformation functions $f_d^{(uf)}(\eta)$, $f_d^{(w)}(\eta)$, $f_d^{(lf)}(\eta)$ are determined with consideration of the classical shear stress formula, called Zhuravsky shear stress, in the form

$$\tau_{xy}^{(Cl)}(x,y) = \frac{S_z(y)}{b(y)}\frac{T(x)}{J_z} \tag{20}$$

where: $S_z(y) = \bar{S}_z(\eta)bh^2$ – first moment of the cross section part about the z-axis, T(x) – transverse-shear force, $J_z = \bar{J}_z bh^3$ – inertia moment of the cross section about the z-axis, $\bar{J}_z$. – dimensionless inertia moment.

Therefore, this formula in the dimensionless coordinate η is as follows

$$\tau_{xy}^{(Cl)}(x,\eta) = \frac{\bar{S}_z(\eta)}{b(\eta)}\frac{T(x)}{h\bar{J}_z} \tag{21}$$

The dimensionless first moments of successive layers of this beam are as follows:

- the upper flange ($-1/2 \le \eta \le -\chi_w/2$)

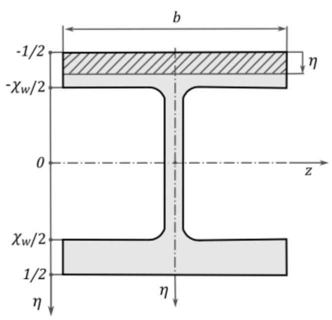


Fig. 4. The hatchet area of the beam cross section selected part

The dimensionless first moment of the hatched area (Fig. 4) is as follows

$$\bar{S}_z^{(uf)}(\eta) = \frac{1}{8}(1-4\eta^2) \tag{22}$$

- the web ($-\chi_w/2 \le \eta \le \chi_w/2$)

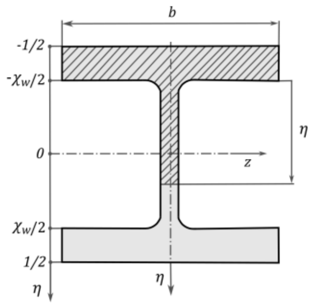


Fig. 5. The hatchet area of the beam cross section selected part

The dimensionless first moment of the hatched area (Fig. 5) is as follows

$$\bar{S}_z^{(w)}(\eta) = \frac{1}{8}(1-\chi_w^2) - J_w(\eta) \tag{23}$$

where $J_w(\eta) = \int_{-\frac{\chi_w}{2}}^{\eta} f_w(\eta_1)d\eta_1$

- the lower flange ($\chi_w/2 \le \eta \le 1/2$)

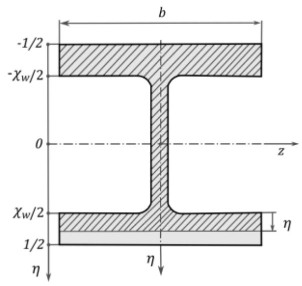


Fig. 6. The hatchet area of the beam cross section selected part

The dimensionless first moment of the hatched area (Fig.6) is as follows

$$\bar{S}_z^{(lf)}(\eta) = \frac{1}{8}(1-4\eta^2) \tag{24}$$

Equating shear stresses (11), (15) and (19) to the classical shear stress formula (21), taking into account dimensionless first moments (22), (23) and (24), after simply transformation one obtains the unknown dimensionless deformation functions of successive layers in the following form:

- the upper flange ($-1/2 \le \eta \le -\chi_w/2$)

$$f_d^{(uf)}(\eta) = -C_f + \frac{1}{24}(3-4\eta^2)\eta \tag{25}$$

- the web ($-\chi_w/2 \le \eta \le \chi_w/2$)

$$f_d^{(w)}(\eta) = \frac{1}{8}\int\frac{1-\chi_w^2-8J_w}{f_w(\eta)}d\eta \tag{26}$$

- the lower flange ($\chi_w/2 \le \eta \le 1/2$)

$$f_d^{(lf)}(\eta) = C_f + \frac{1}{24}(3-4\eta^2)\eta \tag{27}$$

where $C_f = -\frac{1}{48}(3-\chi_w^2)\chi_w + \frac{1}{8}\int_0^{\frac{\chi_w}{2}}\frac{1-\chi_w^2-8J_w}{f_w(\eta)}d\eta$

The elastic strain energy of the beam

$$U_{\varepsilon,\gamma} = \frac{1}{2}\text{Ebh}\int_0^L\left\{\Phi_{\varepsilon,\gamma}^{(uf)}(x,t) + \Phi_{\varepsilon,\gamma}^{(w)}(x,t) + (x,t)\right\}dx \quad (28)$$

where:

$$\Phi_{\varepsilon,\gamma}^{(uf)}(x,t) = \int_{-1/2}^{-\chi_w/2}\left\{\left[\varepsilon_x^{(uf)}(x,\eta,t)\right]^2 + \frac{1}{2(1+\nu)}\left[\gamma_{xy}^{(uf)}(x,\eta,t)\right]^2\right\}d\eta \quad (29)$$

$$\Phi_{\varepsilon,\gamma}^{(w)}(x,t) = \int_{-\chi_w/2}^{\chi_w/2}\left\{\left[\varepsilon_x^{(w)}(x,\eta,t)\right]^2 + \frac{1}{2(1+\nu)}\left[\gamma_{xy}^{(w)}(x,\eta,t)\right]^2\right\}d\eta \quad (30)$$

$$\Phi_{\varepsilon,\gamma}^{(lf)}(x,t) = \int_{\chi_w/2}^{1/2}\left\{\left[\varepsilon_x^{(lf)}(x,\eta,t)\right]^2 + \frac{1}{2(1+\nu)}\left[\gamma_{xy}^{(lf)}(x,\eta,t)\right]^2\right\}d\eta \quad (31)$$

The kinetic energy

$$U_K = \frac{1}{2}\varrho_b\bar{A}bh\int_0^L\left(\frac{\partial v}{\partial t}\right)^2 dx \quad (32)$$

where: $\bar{A} = A/bh$ – dimensionless area cross section, ϱ_b – mass density of the beam.

Based on the Hamilton's principle

$$\delta\int_{t_1}^{t_2}\left(U_K - U_{\varepsilon,\gamma}\right)dt = 0 \quad (33)$$

two differential equations of motion are obtained in the following form:

$$\varrho_b\bar{A}\frac{\partial^2 v}{\partial t^2} + Eh^2\left(\bar{J}_z\frac{\partial^4 v}{\partial x^4} - C_{v\psi}\frac{\partial^3\psi_s}{\partial x^3}\right) = 0 \quad (34)$$

$$C_{v\psi}\frac{\partial^3 v}{\partial x^3} - C_{\psi\psi}\frac{\partial^2\psi_s}{\partial x^2} + C_\psi\frac{\psi_s(x,t)}{h^2} = 0 \quad (35)$$

where: dimensionless coefficients

$$\bar{J}_z = \frac{1}{12}(1-\chi_w^3) + 2\int_0^{\chi_w/2}\eta^2 f_w(\eta)d\eta$$

$$C_{v\psi} = \frac{1}{480}\left[120(1-\chi_w^2)C_f + 4 - 5\chi_w^3 + \chi_w^5\right] + 2\int_0^{\chi_w/2}\eta f_d^{(w)}(\eta)f_w(\eta)d\eta$$

$$C_{\psi\psi} = 2\int_0^{\chi_w/2}\left[f_d^{(w)}(\eta)\right]^2 f_w(\eta)d\eta + 2\int_{\chi_w/2}^{1/2}\left[f_d^{(lf)}(\eta)\right]^2 d\eta$$

$$C_\psi = \frac{1}{64(1+\nu)}\left\{\frac{1}{30}(8 - 15\chi_w + 10\chi_w^3 - 3\chi_w^5) + \int_0^{\chi_w/2}\frac{\left[1-\chi_w^2 - 8J_w(\eta)\right]^2}{f_w(\eta)}d\eta\right\}$$

3. Analytical determination of the fundamental natural frequency

Two differential equations of motion (34) and (35) for the free flexural vibration of the simply supported H-beam are approximately solved with the use of two assumed functions:

$$v(x,t) = v_a(t)\sin\left(\pi\frac{x}{L}\right) \quad (36)$$

$$\psi_s(x,t) = \psi_{sa}(t)\cos\left(\pi\frac{x}{L}\right) \quad (37)$$

where: $v_a(t)$, $\psi_{sa}(t)$ – functions of the time t.

Substituting these functions (36) and (37) into equations (34) and (35), after simply transformation, one obtains the following differential equation

$$\frac{\partial^2 v}{\partial t^2} + (1 - C_{se})\frac{\pi^4\bar{J}_z E}{\lambda^2 L^2\bar{A}\varrho_b}v_a(t) = 0 \quad (38)$$

where: the shear coefficient

$$C_{se} = \frac{\pi^2}{\pi^2 C_{\psi\psi} + \lambda^2 C_\psi}\frac{C_{v\psi}^2}{\bar{J}_z} \quad (39)$$

and λ = L/h – the relative length of the beam.

The equation (38) is also approximately solved with the use of the function

$$v_a(t) = v_o\sin(\omega t) \quad (40)$$

where: v_o [mm] – the amplitude of the flexural vibration, and ω [1/s] – the fundamental natural frequency.

Substituting this function into the equation (38) one obtains the fundamental natural frequency in the following form

$$f_z = \frac{\omega}{2\pi} = \frac{\pi\, 10^6}{2\lambda^2 h}\sqrt{(1 - C_{se})\frac{\bar{J}_z}{\bar{A}}\frac{E}{\varrho_b}} \quad [H_z] \quad (41)$$

where dimensions of quantities: E [MPa], ρ_b [kg/m^3] and length h [mm].

The detailed calculation results for sample standard wide-flange H-beams of following data: λ = 20, ν = 0.3, E = 2 10^5 MPa, ρ_b = 7850 kg/m^3 are specified in Table 1.

Table 1. The selected results of the detailed calculation

Beam	H-100	H-200	H-300
$\bar{A}$	0.26	0.195224	0.16557
$\bar{J}_z$	0.045	0.035624	0.031073
β_0	0.060	0.045	0.036667
α_r	0.219	0.126	0.1177
χ_w	0.80	0.85	0.873333
n	16	16	16
C_{se}	0.0432860	0.0460501	0.0489381
f_z [Hz]	80.660	41.350	27.914

The values of the even exponent n and dimensionless coefficient α_r are determined taking into account the values of the dimensionless area cross section $\bar{A}$ and the dimensionless inertia moment $\bar{J}_z$.

4. Conclusions

Taking into account the analytically determined fundamental natural frequency (41) of the H-beam, with consideration of the shear effect, the following comments formulated:

a) the shear effect (expressed by the coefficient C_{se}) reduced the fundamental natural frequency
b) the value of this coefficient (39) decreases as the relative length of the beam λ increases
c) the values of this coefficient C_{se} for sample standard H-beams (Table 1) are less than 5%, thus, the values of the fundamental natural frequency are reduced by 2.5%
d) the calculation of the fundamental natural frequencies for these beams in the engineering practice can be carried out without the shear effect (Cse = 0) – the classical formula (41).

Acknowledgements

The paper is developed based on the scientific activity of the Łukasiewicz Research Network – Poznan Institute of Technology.

Bibliography

[1] Aghababaei R, Reddy JN. Nonlocal third-order shear deformation plate theory with application to bending and vibration of plates. J Sound Vib. 2009;326:277-289. https://doi.org/10.1016/j.jsv.2009.04.044

[2] Akgöz B, Civalek Ö. A size-dependent shear deformation beam model based on the strain gradient elasticity theory. Int J Eng Sci. 2013;70;1-14. https://doi.org/10.1016/j.ijengsci.2013.04.004

[3] Ghugal YM, Sharma R. A hyperbolic shear deformation theory for flexure and vibration of thick isotropic beams. Int J Comput Methods. 2009;6(4):585-604. https://doi.org/10.1142/S0219876209002017

[4] Guo Q, Shi G. An accurate and efficient 4-noded quadrilateral plate element for free vibration analysis of laminated composite plates using a refined third-order shear deformation plate theory. Compos Struct. 2023;324:117490. https://doi.org/10.1016/j.compstruct.2023.117490

[5] Magnucki K. Bending of symmetrically sandwich beams and I-beams – analytical study. Int J Mech Sci. 2019;150:411-419. https://doi.org/10.1016/j.ijmecsci.2018.10.020

[6] Magnucki K, Lewinski J, Magnucka-Blandzi E. A shear deformation theory of beams with bisymmetrical cross-section based on the Zhuravsky shear stress formula. Engineering Transactions. 2020;68(4):353-370. https://doi.org/10.24423/EngTrans.1174.20201120

[7] Magnucki K. An individual shear deformation theory of beams with consideration of the Zhuravsky shear stress formula. In: Zingoni A (ed.) Current perspectives and new directions in mechanics, modelling and design of structural systems. CRC Press, Taylor & Francis Group. London, New York 2022:682-689. https://doi.org/10.1201/9781003348443-112

[8] Mahi A, Bedia EAA, Tounsi A. A new hyperbolic shear deformation theory for bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates. Appl Math Model. 2015;39:2489-2508. https://doi.org/10.1016/j.apm.2014.10.045

[9] Nguyen T-K, Nguyen TT-P, Vo TP, Thai H-T. Vibration and buckling analysis of functionally graded sandwich beams by a new higher-order shear deformation theory. Compos Part B-Eng. 2015;76:273-285. https://doi.org/10.1016/j.compositesb.2015.02.032

[10] Reddy JN. Nonlocal nonlinear formulations for bending of classical and shear deformation theories of beams and plates. Int J Eng Sci. 2010;48:1507-1518. https://doi.org/10.1016/j.ijengsci.2010.09.020

[11] Ren S, Cheng C, Meng Z, Yu B, Zhao G. A new general third-order zigzag model for asymmetric and symmetric laminated composite beams. Compos Struct. 2021;260:113523. https://doi.org/10.1016/j.compstruct.2020.113523

[12] Sawant M, Dahake AG. A new hyperbolic shear deformation theory for analysis of thick beam. International Journal of Innovative Research in Science, Engineering and Technology. 2014;3(2):9636-9643. https://www.researchgate.net/publication/260366415

[13] Sobhy M. An accurate shear deformation theory for vibration and buckling of FGM sandwich plates in hygrothermal environment. Int J Mech Sci. 2016;110:62-77. http://doi.org/10.1016/j.ijmecsci.2016.03.003

[14] Thai H-T, Vo TP. Bending and free vibration of functionally graded beams using various higher-order shear deformation beam theories. Int J Mech Sci. 2012; 62: 57-66. https://doi.org/10.1016/j.ijmecsci.2012.05.014

[15] Thai S, Thai H-T, Vo TP, Patel VI. A simple shear deformation theory for nonlocal beams. Compos Struct. 2018;183:262-270. https://doi.org/10.1016/j.compstruct.2017.03.022

[16] Xiang S. A new shear deformation theory for free vibration of functionally graded beams. Appl Mech Mater. 2014;455:198-201. https://doi.org/10.4028/www.scientific.net/AMM.455.198